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Solving Fuzzy Nonlinear Equation using Harmonic Mean Method

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Abstract—In this paper, a new algorithm is proposed to solve fuzzy nonlinear equations without converting it to crisp equivalent problem and obtained a fuzzy solution using Harmonic mean method .Instead of bisecting the interval, we divide the interval using Harmonic mean to find the approximate root. A numerical example is provided to illustrate the method proposed in this paper.

Index Terms—Harmonic mean, F(R), the set of triangular fuzzy numbers, Ranking of fuzzy numbers, Fuzzy nonlinear equation(FNLE).

1. INTRODUCTION

THE outcome of a physical system may depends on the roots of algebraic equations. One can use analytical or numerical methods to compute the roots of a non-linear equation to any desired degree of accuracy, where the parameters of the nonlinear equation are represented by crisp numbers. However, in most applications, the parameters of the system of nonlinear equations and measurements are not always represented by crisp numbers but also by fuzzy numbers. Hence, it is necessary to develop an appropriate method for investigating the fuzzy linear systems.

Subhash and Sathya [11] proposed a method using linear interpolation to solve a non linear equation f(x) =0, which is a modification of fuzzy Newton-Raphson method that converges rapidly to exact solution. Gautam and Shirin [4] proposed a method to solve a fuzzy non-linear equation with the help of Bisection Algorithm. Abbasbandy and Asady [1] discussed fuzzy nonlinear equations using Newton's method. Javad Shokri [5] studied fuzzy nonlinear equations using numerical methods. Mahmoud Paripour et.al [7] discussed numerical solution for a system of fuzzy nonlinear equations. Khorasani and Aghcheghloo [10] studied fuzzy nonlinear Equation using Secand Method. Waziri et.al [12] proposed a new approach for solving dual fuzzy nonlinear equations using Broyden's and Newton's methods.

In this paper, we propose Harmonic mean algorithm for the solution of fuzzy nonlinear equations without converting to crisp form.

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The present paper is organized as follows: In Section 2, some preliminary results related to F(R) are given. Using a new representation of fuzzy numbers proposed by Ming Ma et .al [6] the arithmetic operations are defined. Ranking of fuzzy numbers [7] are defined based on this new representation of fuzzy numbers. Section 3 deals with an algorithm to solve nonlinear equation based on bisection method. Numerical examples are given in Section 4 to illustrate the proposed method. Conclusion of this paper is given in Section 5.

2. PRELIMINARIES

The some basic concepts and related results of triangular fuzzy numbers are recalled.

Definition 2.1 A fuzzy set $\tilde{a}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function $\tilde{a}: R \to [0,1]$ has the following characteristics:

(i) $\tilde{a}$ is convex

i.e. $\tilde{a}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{a}(x_1),\ \tilde{a}(x_2)\}$, for all $x_1, x_2 \in R$ and $\lambda \in [0,1]$.

(ii) $\tilde{a}$ is normal i.e., there exists an $x \in R$ such that $\tilde{a}(x) = 1$

(iii) $\tilde{a}$ is Piecewise continuous.

Definition 2.2 A fuzzy number $\tilde{a}$ on R is said to be a triangular fuzzy number F(R) or linear real fuzzy number(LFRN) if its membership function $\mu : R \to [0,1]$ has the following characteristics:

$$\mu(x) = \begin{cases} 0, \text{if } x \leq a \\ \dfrac{x-a}{b-a}, \text{if } a \leq x \leq b \\ \dfrac{c-x}{c-b}, \text{if } b \leq x \leq c \end{cases} \quad (1)$$

The triangular fuzzy number is denoted by $\tilde{a}$ = (a, b, c). We use F(R) denote the set of all triangular fuzzy numbers. Also if $m = b$ represents the modal value or midpoint, $\alpha = (b-a)$ represents the left spread and $\beta = (c-b)$ represents the right spread of the triangular fuzzy number $\tilde{a}$ = (a, b, c) then the triangular fuzzy number $\tilde{a}$ can also be represented by the triplet $\tilde{a} = (\alpha, m, \beta)$. i.e., $\tilde{a} = (a, b, c) = (\alpha, m, \beta)$.

The concept of linear real fuzzy number is discussed [4]. Here it is noted that any real number b can be written as a linear fuzzy real number (b, b, b) and so $R \subseteq F(R)$.

Definition 2.3. A triangular fuzzy number $\tilde{a} \in F(R)$ can also be represented as a pair $\tilde{a} = (\underline{a}, \overline{a})$ of functions $\underline{a}(r)$ and $\overline{a}(r)$ for $0 \le r \le 1$ which satisfies the following requirements:

(i). $\underline{a}(r)$ is a bounded monotonic increasing left continuous function.

(ii). $\overline{a}(r)$ is a bounded monotonic decreasing left continuous function.

(iii). $\underline{a}(r) \le \overline{a}(r)$, $0 \le r \le 1$

Definition 2.4. For an arbitrary triangular fuzzy number $\tilde{a} = (\underline{a}, \overline{a})$, the number $a_0 = \left(\frac{\underline{a}(1)+\overline{a}(1)}{2}\right)$ is said to be a location index number of $\tilde{a}$. The two non-decreasing left continuous functions $a_* = (a_0 - \underline{a})$, $a^* = (\overline{a} - a_0)$ are called the left fuzziness index function and the right fuzziness index function respectively. Hence every triangular fuzzy number $\tilde{a} = (a, b, c)$ can also be represented by $\tilde{a} = (a_0, a_*, a^*)$.

2.1. Ranking of triangular fuzzy numbers

Many different approaches for the ranking of fuzzy numbers have been proposed in the literature. For an arbitrary triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ $= (a_0, a_*, a^*)$ with parametric form $\tilde{a} = (\underline{a}(r), \overline{a}(r))$, the magnitude of the triangular fuzzy number $\tilde{a}$ is defined by:

$$Mag(\tilde{a}) = \frac{1}{2}\int_0^1 (\underline{a} + \overline{a} + a_0) f(r) dr \qquad (2)$$

where the function $f(r)$ is a non-negative and increasing function on $[0,1]$ with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)\,dr = \frac{1}{2}$.

The function $f(r)$ can be considered as a weighting function. In real life applications, $f(r)$ can be chosen by the decision maker according to the situation. In this paper, for convenience $f(r) = r$. Hence equation (2) becomes

$$Mag(\tilde{a}) = \frac{a^* + 4a_0 - a_*}{6} \qquad (3)$$

The magnitude of a triangular fuzzy number $\tilde{a}$ synthetically reflects the information on every membership degree, and meaning of this magnitude is visual and natural. $Mag(\tilde{a})$ is used to rank fuzzy numbers. The larger $Mag(\tilde{a})$, the larger fuzzy number.

For any two triangular fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ and $\tilde{b} = (b_0, b_*, b^*)$ in F(R), define the ranking of $\tilde{a}$ and $\tilde{b}$ by comparing the $Mag(\tilde{a})$ and $Mag(\tilde{b})$ on R as follows:

(i). $\tilde{a} \succeq \tilde{b}$ if and only if $Mag(\tilde{a}) \ge Mag(\tilde{b})$

(ii). $\tilde{a} \preceq \tilde{b}$ if and only if $Mag(\tilde{a}) \le Mag(\tilde{b})$ ≥

(iii). $\tilde{a} \approx \tilde{b}$ if and only if $Mag(\tilde{a}) = Mag(\tilde{b})$

Definition 2.5. A triangular fuzzy number $\tilde{a} = (a_0, a_*, a^*)$ is said to be symmetric if and only if $a_* = a^*$.

Definition 2.6. A triangular fuzzy number $\tilde{a} = (a_0, a_*, a^*)$ is said to be non-negative if and only if $Mag(\tilde{a}) \ge 0$ and is denoted by $\tilde{a} \succeq \tilde{0}$. Further if $Mag(\tilde{a}) > 0$, then $\tilde{a} = (a_0, a_*, a^*)$ is said to be a positive fuzzy number and is denoted by $\tilde{a} \succ \tilde{0}$.

Definition 2.7. Two triangular fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ and $\tilde{b} = (b_0, b_*, b^*)$ in $F(R)$ are said to be equivalent if and only if $Mag(\tilde{a}) = Mag(\tilde{b})$. That is $\tilde{a} \approx \tilde{b}$ if and only if $Mag(\tilde{a}) = Mag(\tilde{b})$. Two triangular fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ and $\tilde{b} = (b_0, b_*, b^*)$ in $F(R)$ are said to be equal if and only if $a_0 = b_0, a_* = b_*, a^* = b^*$. that is $\tilde{a} = \tilde{b}$ if and only if $a_0 = b_0, a_* = b_*, a^* = b^*$.

Definition 2.8. The Harmonic mean of two numbers a and b is defined as $HM = \frac{2ab}{a+b}$

2.2. Arithmetic operation on triangular fuzzy numbers

Ming Ma et al. [6] have proposed a new fuzzy arithmetic based upon both location index and fuzziness index functions. The location index number is taken in the ordinary arithmetic, whereas the fuzziness index functions are considered to follow the lattice rule which is least upper bound in the lattice L. That is for a, b∈L, define

$$a \vee b = \max\{a, b\}, a \wedge b = \min\{a, b\}$$

For arbitrary triangular fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ and $\tilde{b} = (b_0, b_*, b^*)$ and $* = \{+, -, \times, \div\}$, the arithmetic operations on the triangular fuzzy numbers are defined by $\tilde{a} * \tilde{b} = (a_0 * b_0,\ a_* \vee b_*,\ a^* \vee b^*)$.

In particular for any two triangular fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ and $\tilde{b} = (b_0, b_*, b^*)$, define

(i). Addition : $\tilde{a} + \tilde{b} = (a_0, a_*, a^*) + (b_0, b_*, b^*)$

$$= \left(a_0 + b_0, \max\{a_*, b_*\}, \max\{a^*, b^*\}\right)$$

(ii). Subtraction : $\tilde{a} - \tilde{b} = (a_0, a_*, a^*) - (b_0, b_*, b^*)$

$$= \left(a_0 - b_0, \max\{a_*, b_*\}, \max\{a^*, b^*\}\right).$$

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(iii). Multiplication : $\tilde{a}\times\tilde{b}=\left(a_0,a_*,a^*\right)\times\left(b_0,b_*,b^*\right)$

$$=\left(a_0\times b_0,\max\left\{a_*,b_*\right\},\max\left\{a^*,b^*\right\}\right).$$

(iv). Division : $\tilde{a}\div\tilde{b}=\left(a_0,a_*,a^*\right)\div\left(b_0,b_*,b^*\right)$

$$=\left(a_0\div b_0,\max\left\{a_*,b_*\right\},\max\left\{a^*,b^*\right\}\right).$$

3. FUZZY NON-LINEAR ALGEBRAICEQUATION

A non-linear equation over linear fuzzy real numbers is called a fuzzy non-linear equation. Here, it is converted into a new representation proposed by Ming Ma [6].For this conversion, the first concern is the integral power of triangular fuzzy number. It is discussed in [4] and it can be obtained in terms of location index function and two fuzziness functions by the following lemma.

Lemma 3.1. $(a,b,c)^n$ can be represented in terms of location index function and two fuzziness functions.

Proof: It is proved this by induction on n.

For n=2,

$$(a,b,c)^2=(a,b,c).(a,b,c)$$
$$=(b_0,(b-a)(1-r),(c-b)(1-r))$$
$$\times(b_0,(b-a)(1-r),(c-b)(1-r))$$
$$=(b_0{}^2,(b-a)(1-r),(c-b)(1-r))$$

By induction,

$$(a,b,c)^n=(b_0{}^n,(b-a)(1-r),(c-b)(1-r)) \qquad (4)$$

Hence the lemma.

Consider a fuzzy non-linear equation

$$\tilde{a}_0\tilde{x}^n+\tilde{a}_1\tilde{x}^{n-1}+\ldots+\tilde{a}_n=\tilde{0} \qquad (5)$$

where $\tilde{a}_n=(a_n,b_n,c_n)$ and

$\tilde{x}=(x,y,z)$

Then the above polynomial can be represented in terms of left and right index function by definition 2.4 and by lemma 2.1.

$$\tilde{f}(\tilde{x})=(b_0,\alpha_{0^*},\beta^{0^*})\,(y^n,\alpha_*,\beta^*)+(b_1,\alpha_{1^*},\beta^{1^*})$$
$$(y^{n-1},\alpha_*,\beta^*)+\ldots..+(b_1,\alpha_{(n-1)^*},\beta^{(n-1)^*})(y,\alpha_*,\beta^*)$$
$$+(b_n,\alpha_{n^*},\beta^{n^*}),\text{ where},\alpha_*=(y-x)(1-r),$$
$$\beta^*=(z-y)(1-r),\ \alpha_{n^*}=(b_n-a_n)(1-r)\text{ and}$$
$$\beta^{n^*}=(c_n-b_n)(1-r) \qquad (6)$$
$$\tilde{f}(\tilde{x})=\left(b_0\,y^n,\max\{\alpha_{0^*},\alpha_*\},\max\{\beta^{0^*},\beta^*\}\right)+(b_0\,y^{n-1},$$
$$\max\{\alpha_{1^*},\alpha_*\}),\max\{\beta^{1^*},\beta^*\}+\ldots.+(b_n,\alpha_{n^*},\beta^{n^*})$$
$$=\left(\sum_{i=0}^{n}b_iy^{n-i},\max_{0\le i\le n}\{\alpha_{i^*},\alpha_*\},\max_{0\le i\le n}\{\beta^{i^*},\beta^*\}\right)$$

By the arithmetic operation defined in section 2

3.1. Harmonic Mean Method to Solve Fuzzy Non-Linear Equation

Consider a fuzzy nonlinear algebraic equation (6)

$\tilde{f}(\tilde{x})=\tilde{0}$

Let $\tilde{x}_1=(m_1,\alpha_{1^*},\beta_{1^*})$ and $\tilde{x}_2=(m_2,\alpha_{2^*},\beta_{2^*})$ be two initial approximations. Then

$$\tilde{f}(\tilde{x}_1)=\left(\sum_{i=0}^{n}b_im_1{}^{n-i},\max_{0\le i\le n}\{\alpha_{i^*},\alpha_{1(xy)^*}\},\max_{0\le i\le n}\{\beta^{i^*},\beta^{1(yz)^*}\}\right)$$
$$\tilde{f}(\tilde{x}_2)=\left(\sum_{i=0}^{n}b_im_2{}^{n-i},\max_{0\le i\le n}\{\alpha_{i^*},\alpha_{2(xy)^*}\},\max_{0\le i\le n}\{\beta^{i^*},\beta^{2(yz)^*}\}\right)$$

where $\alpha_{1(xy)^*}=(m_1-x_1)(1-r),\ \beta^{1(yz)^*}=(z_1-y_1)(1-r)$

$\alpha_{2(xy)^*}=(y_2-x_2)(1-r),\beta^{2(yz)^*}=(z_2-y_2)(1-r)$

If $mag(\tilde{f}(\tilde{x}_1))$ and $mag(\tilde{f}(\tilde{x}_2))$ are of opposite signs, then there exists a root between $\tilde{x}_1$ and $\tilde{x}_2$...

Let it be $\tilde{x}_3$ and $\tilde{x}_3=\dfrac{2\tilde{x}_1\tilde{x}_2}{\tilde{x}_1+\tilde{x}_2}$. If $mag(\tilde{f}(\tilde{x}_3))$ and $mag(\tilde{f}(\tilde{x}_1))$ are of opposite signs, then there exists a root between $\tilde{x}_3$ and $\tilde{x}_1$. Let it be $\tilde{x}_4$ and $\tilde{x}_4=\dfrac{2\tilde{x}_1\tilde{x}_3}{\tilde{x}_1+\tilde{x}_3}$. On the other hand if $mag(\tilde{f}(\tilde{x}_2))$ and $mag(\tilde{f}(\tilde{x}_3))$ are of opposite signs, then there exists a root between $\tilde{x}_2$ and $\tilde{x}_3$. Let it be $\tilde{x}_4$ and

$\tilde{x}_4=\dfrac{2\tilde{x}_2\tilde{x}_3}{\tilde{x}_2+\tilde{x}_3}$. Proceeding in this way, a sequence of roots $\{\tilde{x}\}$ can be obtained and it converges to the exact root.

4. NUMERICAL EXAMPLE

Consider a fuzzy nonlinear equation $\tilde{x}^3+\tilde{x}^2-\tilde{1}=0$ discussed in [2]. Let $\tilde{x}_1=(0,0.5,1.0)$ and $\tilde{x}_2=(0.5,1.0,1.5)$ be the two initial roots of the equation. Then $\tilde{x}_1=(0.5,0.5(1-r),0.5(1-r)),\ \tilde{x}_2=(1.0,0.5(1-r),0.5(1-r))$.

Then from (6), we have $\tilde{f}(\tilde{x}_1)=(-0.625,0.5(1-r),0.5(1-r))$ and $\tilde{f}(\tilde{x}_2)=(1.0,0.5(1-r),0.5(1-r))$.

Now $Mag(\tilde{f}(\tilde{x}_1))<0$ and $Mag(\tilde{f}(\tilde{x}_2))>0$. Then root lies between $\tilde{x}_1$ and $\tilde{x}_1$. Hence

$$\tilde{x}_3=\frac{2\tilde{x}_1\tilde{x}_2}{\tilde{x}_1+\tilde{x}_2}=(0.667,0.5(1-r),0.5(1-r))$$

Proceeding in this way, a sequence of roots $\{\tilde{x}\}$ can be obtained.

Table 1. Roots of the polynomial

S.No	Roots	
1	μ_3= (0.6667, 0.5(1-r), 0.5(1-r))	Mag f(μ_3) < 0
2	μ_4= (0.80002, 0.5(1-r), 0.5(1-r))	Mag f(μ_4) > 0
3	μ_5= (0.7273, 0.5(1-r), 0.5(1-r))	Magf(μ_5) < 0
4	μ_6= (0.76193, 0.5(1-r), 0.5(1-r))	Magf(μ_6) > 0
5	μ_7= (0.74421, 0.5(1-r), 0.5(1-r))	Magf(μ_7) < 0
6	μ_8= (0.75297, 0.5(1-r), 0.5(1-r))	Magf(μ_8) < 0
7	μ_9= (0.75742, 0.5(1-r), 0.5(1-r))	Magf(μ_9) > 0
8	μ_{10} = (0.75519, 0.5(1-r), 0.5(1-r))	

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5. CONCLUSION

Based on the new arithmetic defined in [6] and ranking method [7], Fuzzy polynomial is converted into a new parametric form. With help of proposed Harmonic mean algorithm, root of this polynomial is found. The numerical results are given in the above table. It is to be noted that this method has the flexibility in choosing $r \in [0,1]$. particularly for r=1, crisp solution for equation (6) can be obtained.

REFERENCES

[1]. Abbasbandy S., Asady B."Newton's method for solving fuzzy nonlinear equations,"Appl. Math. Comput, 159 (2004) 349-356.

[2]. Abbasbandy. S and Hajjari. T, A new approach for ranking of trapezoidal fuzzy numbers, Computers and Mathematics with Applications, 57 (2009), 413–419.

[3]. Bashar M.A and S.Shirin , Squares and Square roots of continuous fuzzy numbers, Dhakauniv.J.Sci, 53(2), July 2005, 131-140

[4]. Goutam Kumar Saha and Shapla Shirin, Solution of Fuzzy Non-linear Equation using Bisection Algorithm, Dhaka Univ. J. Sci. 61(1),2013, 53-58.

[5]. Javad Shokri, Numerical Method for Solving Fuzzy Nonlinear Equations,Applied Mathematical Sciences, Vol. 2, 2008, no. 24, 1191 – 1203

[6]. Ming Ma, Menahem Friedman and Abraham Kandel, A new fuzzy arithmetic. Fuzzy sets and systems.108 (1999) 83-90.

[7]. Mahmoud Paripour , Eisa Zarei, Ahmad Shahsavaran, Numerical solution for a system of fuzzy nonlinear equations, Journal of Fuzzy Set Valued Analysis 2014 (2014) 1-10

[8]. J. Neggers, H. KIM, On Linear Fuzzy Real numbers, Manuscript for book under development, 2007

[9]. Rogers, F., Jun, Y., Fuzzy Nonlinear Optimization for the Linear Fuzzy Real Number System, International Mathematical Forum, 4(12), 2009, 589 – 596.

[10]. Seied Mahmoud Khorasani and M. H. Dogani Aghcheghloo, Solving Fuzzy Nonlinear Equation with Secand Method, International Journal of Algebra, Vol. 5, 2011, no. 6, 295 – 299

[11]. Subash, R. and Sathya, Numerical solution of fuzzy modified Newton-Raphson method for solving nonlinear equations, International Journal of Current Research, 3(11), 2011, 390-392.

[12]. M. Y. Waziri, and Z. A. Majid, A new approach for solving dual fuzzy nonlinear equations using Broyden's and Newton's methods, Advances in Fuzzy Systems, Volume 2012, Article No. 19 .

[13]. L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.